Nonequilibrium dynamics of urn models

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Abstract
Dynamical urn models, such as the Ehrenfest model, played an important role in
the early days of statistical mechanics. Dynamical many-urn models generalize
the former models in two respects: the number of urns is macroscopic, and
thermal effects are included. These many-urn models are exactly solvable in the
mean-field geometry. They allow analytical investigations of the characteristic
features of nonequilibrium dynamics referred to as aging, including the scaling
of correlation and response functions in the two-time plane and the violation of
the fluctuation-dissipation theorem. This paper contains a general presentation
of these models, as well as a more detailed description of two dynamical urn
models, the backgammon model and the zeta urn model.

Prologue

Urns containing balls, like dice or playing cards, are ubiquitous in writings on probability
theory, reminding us that this branch of mathematics owes its early developments to practical
questions arising in playing games. Dynamical urn models, such as the Ehrenfest model, have
played an important role in the elucidation of conceptual problems in statistical mechanics.
More recently, dynamical many-urn models have been investigated in the mean-field geometry.
These exactly solvable models exhibit characteristic features of nonequilibrium dynamics
referred to as aging, such as the scaling of the correlation and response functions in the two-
time plane and the violation of the fluctuation-dissipation theorem.

    This paper contains a didactic introduction to urn models (section 1), a presentation of static
and dynamical properties of many-urn models (section 2), a reminder of the main characteristic
features of aging (section 3) and an overview of recent results on two dynamical urn models,
namely the backgammon model (section 4) and the zeta urn model (section 5).

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1. Urn models

1.1. The Ehrenfest urn model

This model was devised by P and T Ehrenfest in 1907 [1] in their attempt to critically review Boltzmann's H-theorem. It is defined as follows: N balls are distributed in two urns (or boxes). At random times, given by a Poisson process with unit rate, a ball is chosen at random, and moved from the box in which it is to the other box.

Let $N_1(t)$ ($N_2(t) = N - N_1(t)$) be the number of balls in box number 1 (number 2) at time t. For any initial configuration, the system relaxes to equilibrium for infinitely long times. The equilibrium state is characterized by a binomial distribution of the number N_1:

$$f_{k,\mathrm{eq}} = \mathcal{P}_{\mathrm{eq}}(N_1 = k) = 2^{-N}\binom{N}{k} \qquad (k = 0, \ldots, N) \tag{1.1}$$

reminiscent of the Maxwell–Boltzmann statistics for indistinguishable particles.

As noticed first by Kohlrausch and Schrödinger [2], $N_1(t)$ can be viewed as the co-ordinate of a fictitious walker. The temporal evolution of the occupation probabilities $f_k(t) = \mathcal{P}(N_1(t) = k)$ is governed by the master equation

$$\frac{\mathrm{d}f_k(t)}{\mathrm{d}t} = \frac{k+1}{N}f_{k+1}(t) + \frac{N+1-k}{N}f_{k-1}(t) - f_k(t). \tag{1.2}$$

Indeed, a move of a ball from box number 1 to box number 2 (from box number 2 to box number 1) yields $k \to k-1$ ($k \to k+1$), and occurs with a rate k/N (($N-k$)/N) per unit time. The master equation (1.2) describes a nonuniformly biased random walk over the integers $0, \ldots, N$ [2, 3]. The first (the second) term in the right-hand side is absent for $k = N$ ($k = 0$).

The spectrum of relaxation rates of the model can be derived by looking for solutions to (1.2) of the form $f_k(t) = \phi_k\,\mathrm{e}^{-\lambda t}$. One thus obtains $\lambda_m = 2m/N$, with $m = 0, \ldots, N$. The static solution ($m = 0$) is nothing but the equilibrium distribution (1.1). The occupation probabilities have an exponential convergence to their equilibrium values, with a relaxation time

$$\tau_{\mathrm{eq}} = 1/\lambda_1 = N/2.$$

There are, however, much larger timescales in the Ehrenfest model. Consider indeed the length of time it takes for box number 1 to become empty, i.e. the first time t_0 such that $N_1(t_0) = 0$. This time depends on the initial state and on the whole history of the system. Its mean value $T_0 = \langle t_0 \rangle$ can be evaluated as follows. The equilibrium probability for box number 1 to be empty, $f_{0,\mathrm{eq}} = 2^{-N}$, is exponentially small in the number of balls. It is therefore expected that the typical time needed to reach this very rare event scales as

$$T_0 \approx 1/f_{0,\mathrm{eq}} = 2^N. \tag{1.3}$$

(This result can be derived in a more rigorous way. Related matters are discussed in [4, 5].) The process of emptying one of the boxes is therefore characterized by an exponentially large time T_0. Equation (1.3) can be recast as an Arrhenius-like law:

$$T_0 \sim \mathrm{e}^{S_0} \tag{1.4}$$

where $S_0 = N \ln 2$ is the entropy difference between the equilibrium state of the model, where each box contains one half of the balls, up to fluctuations, and the configuration where box number 1 is empty. This entropy difference plays the role of a reduced activation energy in (1.4). This is an elementary example of an entropy barrier.

1.2. The monkey urn model

Let us introduce a variant of the Ehrenfest urn model, which we choose to call the monkey urn model, because it corresponds to the image of a monkey playing at exchanging balls between two boxes. The key difference with the previous case is that now at each time step a box (either 1 or 2) is chosen at random, and one ball is moved from the chosen box (provided it is nonempty) to the other one. The choice of a box, instead of a ball, induces drastic changes in the statics and dynamics of the model.

The equilibrium state of the monkey urn model is now characterized by uniform occupation probabilities

$$f_{k,\mathrm{eq}} = \mathcal{P}_{\mathrm{eq}}(N_1 = k) = \frac{1}{N+1} \qquad (k = 0, \ldots, N). \tag{1.5}$$

The evolution of the occupation probabilities is governed by the master equation

$$\frac{\mathrm{d} f_k(t)}{\mathrm{d}t} = \frac{1}{2} \left(f_{k+1}(t) + f_{k-1}(t) - 2 f_k(t) \right)$$

describing a usual random walk over the integers $0, \ldots, N$, with appropriate reflecting boundary conditions.

The relaxation rates of the model read $\lambda_m = 2 \sin^2(m\pi/[2(N+1)])$, with $m = 0, \ldots, N$. The relaxation time reads therefore

$$\tau_{\mathrm{eq}} = \frac{1}{\lambda_1} \approx \frac{2}{\pi^2} N^2 \tag{1.6}$$

where the scaling $\tau_{\mathrm{eq}} \sim N^2$ originates in the diffusive nature of the motion of balls.

Because of the flatness of the equilibrium distribution (1.5), there are no entropy barriers in the monkey urn model. In particular, the mean time T_0 for a box to become empty scales as N^2, proportionally to the relaxation time (1.6).

2. Many-urn models: statics and dynamics

We now turn to the discussion of static and dynamical properties of two classes of many-urn models, generalizing the Ehrenfest and the monkey urn models described so far [6]. This classification encompasses examples of urn models studied in recent years. The backgammon model [7] is a prototype of the Ehrenfest class, while model B of [8, 9] and the zeta urn model [6, 9] belong to the monkey class.

A many-urn model consists of N balls, distributed among M boxes. Box number i contains N_i balls, with

$$\sum_{i=1}^{M} N_i = N.$$

Most investigations of many-urn models have dealt with the thermodynamic limit ($M \to \infty$, $N \to \infty$), at a fixed density $\rho = N/M$ of balls per box.

2.1. Statics

Two ingredients are necessary in order to define static properties of many-urn models within the framework of statistical mechanics.

A priori statistics. To each configuration $\{N_i\}$ is attributed an *a priori* statistical weight

$$W(\{N_i\}) = \begin{cases} \displaystyle\prod_{i=1}^{M} \frac{1}{N_i!} & \text{(Ehrenfest class)} \\ 1 & \text{(monkey class).} \end{cases} \tag{2.1}$$

The formula for the Ehrenfest class generalizes the binomial law (1.1), and corresponds to the Maxwell–Boltzmann prescription for indistinguishable particles. The formula for the monkey class generalizes the flat distribution (1.5), and formally corresponds to Bose–Einstein statistics for the occupation numbers of quantum states [10].

Hamiltonian. The Hamiltonian of a many-urn model is defined as the sum of independent contributions of boxes, of the form

$$\mathcal{H}(\{N_i\}) = \sum_{i=1}^{M} E(N_i) \tag{2.2}$$

so that the unnormalized Boltzmann weight of a configuration $\{N_i\}$ at temperature $T = 1/\beta$ is

$$e^{-\beta \mathcal{H}(\{N_i\})} = \prod_{i=1}^{M} p_{N_i} \tag{2.3}$$

with

$$p_{N_i} = e^{-\beta E(N_i)}. \tag{2.4}$$

The canonical partition function thus reads

$$Z(M, N) = \sum_{\{N_i\}} W(\{N_i\}) \, e^{-\beta \mathcal{H}(\{N_i\})} \, \delta\left(\sum_i N_i, N\right).$$

Using (2.1) and (2.3), and a contour-integral representation of the Kronecker symbol, we are left with

$$Z(M, N) = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i} z^{N+1}} [P(z)]^M \tag{2.5}$$

where

$$P(z) = \sum_{k \geqslant 0} p_k z^k \times \begin{cases} 1/k! & \text{(Ehrenfest class)} \\ 1 & \text{(monkey class).} \end{cases} \tag{2.6}$$

Static properties of many-urn models are therefore entirely encoded in the temperature-dependent generating series $P(z)$ of the Boltzmann weights p_k of (2.4). In the thermodynamic limit, at fixed density ρ, the free energy per box,

$$\beta F = -\lim_{M \to \infty} \frac{1}{M} \ln Z(M, N) \qquad (N \approx M\rho)$$

can be obtained by evaluating the contour integral in (2.5) by the saddle-point method. The saddle-point value z_s depends on temperature and density through the equation

$$\frac{z_s P'(z_s)}{P(z_s)} = \rho \tag{2.7}$$

and the free energy per box reads

$$\beta F = \rho \ln z_s - \ln P(z_s). \tag{2.8}$$

The equilibrium occupation probabilities $f_{k,\text{eq}}$ of box number i do not depend on the box under consideration. (In the following we shall refer to box number 1 for definiteness.) We have

$$f_{k,\text{eq}} = \frac{p_k z_s^k}{P(z_s)} \times \begin{cases} 1/k! & \text{(Ehrenfest class)} \\ 1 & \text{(monkey class).} \end{cases} \tag{2.9}$$

The above results further simplify at infinite temperature, where $p_k = 1$, independently of the form of the energy function. Equations (2.5)–(2.9) become

$$P(z) = \mathrm{e}^z \qquad Z(M, N) = \frac{M^N}{N!} \qquad z_s = \rho$$

$$\beta F = \rho \ln \rho - \rho \qquad f_{k,\text{eq}} = \mathrm{e}^{-\rho} \frac{\rho^k}{k!} \tag{2.10}$$

for the Ehrenfest class, and

$$P(z) = \frac{1}{1-z} \qquad Z(M, N) = \frac{(M+N-1)!}{N!(M-1)!} \qquad z_s = \frac{\rho}{\rho+1}$$

$$\beta F = \rho \ln \rho - (\rho+1) \ln(\rho+1) \qquad f_{k,\text{eq}} = \frac{\rho^k}{(\rho+1)^{k+1}} \tag{2.11}$$

for the monkey class.

2.2. Dynamics

An elementary step of the dynamics consists in moving a ball from a departure box d to an arrival box a. A move is proposed to the system at every time step $\delta t = 1/M$. The precise definition of a dynamical many-urn model should state the following points:

Geometry. The departure box d and the arrival box a have to be connected to each other. Most investigations of dynamical urn models have dealt with the mean-field geometry, where any two boxes are connected to each other.

Dynamical urn models can also be defined on regular lattices in finite dimension, with the convention that urns are located at the sites of the lattice, and that only nearest neighbours are connected.

A priori dynamical rule. The proposed moves are chosen according to some *a priori* dynamical rule. There are three natural rules [8].

- *Rule A (ball–box).* Choose a ball (this defines the departure box d). Choose an arrival box a, connected to d.
- *Rule B (box–box).* Choose a departure box d. Choose uniformly a ball in d. Choose an arrival box a, connected to d.
- *Rule C (ball–ball).* Choose a first ball (this defines the departure box d). Choose a second ball (this defines the arrival box a). Propose the move if the boxes d and a are connected.

The urn models described in section 1 suggest what should be the right relationship between *a priori* statistics and dynamical rule. It indeed turns out that rule A corresponds to the Ehrenfest class, while rule B corresponds to the monkey class. Rule C corresponds to yet another class of dynamical urn models, with no obvious static counterpart [8].

Thermal rule.   The following thermal rules respect detailed balance.

- *Metropolis.* A proposed move is performed with probability

$$\min(1, \exp(-\beta \Delta \mathcal{H})) \tag{2.12}$$

  where $\Delta \mathcal{H} = \mathcal{H}(\{N_i'\}) - \mathcal{H}(\{N_i\})$ is the energy difference between the configurations $\{N_i\}$ before and $\{N_i'\}$ after the proposed move.

- *Heat-bath.* Among all the possible *a priori* choices for the arrival box a, one is chosen with an appropriately normalized probability, proportional to the Boltzmann weight $\exp(-\beta \mathcal{H}(\{N_i'\}))$ of the configuration after the move.

Throughout the following, we restrict ourselves to the mean-field geometry, and to a homogeneous disordered initial state, such as the infinite-temperature equilibrium state (2.10), (2.11).

The key quantities are again the occupation probabilities $f_k(t) = \mathcal{P}(N_1(t) = k)$. Their evolution is given by the master equation

$$\frac{\mathrm{d} f_k(t)}{\mathrm{d}t} = \sum_{\ell \geqslant 0} \left(\pi_{k+1,\ell}(t) + \pi_{\ell,k-1}(t) - \pi_{k,\ell}(t) - \pi_{\ell,k}(t) \right) \tag{2.13}$$

where $\pi_{k,\ell}(t)$ denotes the contribution of a move from the departure box d, containing $N_d = k$ balls, to the arrival box a, containing $N_a = \ell$ balls.

The *a priori* dynamical rule yields

$$\pi_{k,\ell}(t) = f_k(t) f_\ell(t) W_{k,\ell}(t) (1 - \delta_{k,0}) \times \begin{cases} k & \text{(Ehrenfest class)} \\ 1 & \text{(monkey class).} \end{cases}$$

The factor $(1 - \delta_{k,0})$ accounts for the fact that an actual move only takes place if the departure box d is not empty. The acceptance rates $W_{k,\ell}(t)$ depend on the thermal part of the rule. With the notation (2.4), the rates for the Metropolis rule [9],

$$W_{k,\ell} = \min \left(1, \frac{p_{k-1} p_{\ell+1}}{p_k p_\ell} \right)$$

are independent of time, while those for the heat-bath rule [9],

$$W_{k,\ell}(t) = \frac{p_{\ell+1}}{p_\ell} \left(\sum_{m \geqslant 0} f_m(t) \frac{p_{m+1}}{p_m} \right)^{-1}$$

only depend on the label ℓ of the arrival box.

In analogy with (1.2), (2.13) can be recast as the master equation of a random walk over the positive integers ($k = 0, 1, \ldots$):

$$\frac{\mathrm{d} f_k(t)}{\mathrm{d}t} = \mu_{k+1}(t) f_{k+1}(t) + \lambda_{k-1}(t) f_{k-1}(t) - \left(\mu_k(t) + \lambda_k(t) \right) f_k(t) \qquad (k \geqslant 1) \tag{2.14a}$$

$$\frac{\mathrm{d} f_0(t)}{\mathrm{d}t} = \mu_1(t) f_1(t) - \lambda_0(t) f_0(t) \tag{2.14b}$$

where

$$\lambda_k(t) = \frac{1}{f_k(t)} \sum_{\ell \geqslant 0} \pi_{\ell,k}(t) \qquad \mu_k(t) = \frac{1}{f_k(t)} \sum_{\ell \geqslant 0} \pi_{k,\ell}(t) \tag{2.15}$$

are, respectively, the hopping rate to the right, corresponding to $a = 1$, i.e. $N_1 = k \to N_1 = k + 1$, and to the left, corresponding to $d = 1$, i.e. $N_1 = k \to N_1 = k - 1$. The equation for $f_0(t)$ is special, as $\lambda_{-1}(t) = \mu_0(t) = 0$.

The master equation (2.14) preserves the sum rules

$$\sum_{k\geqslant 0} f_k(t) = 1 \qquad \sum_{k\geqslant 0} k\, f_k(t) = \langle N_1(t)\rangle = \rho \qquad\qquad (2.16)$$

expressing the conservation of probability and of the number of balls.

The rates $\lambda_k(t)$ and $\mu_k(t)$, defined in (2.15), depend on the occupation probabilities $f_k(t)$ themselves, so that the master equation (2.14) is actually a nonlinear set of first-order equations for the $f_k(t)$. The equilibrium occupation probabilities (2.9) are recovered as their unique stationary state.

3. Equilibrium versus nonequilibrium dynamics: two-time quantities

Nonequilibrium properties of dynamical urn models such as the backgammon model [7, 8, 11, 12] or the zeta urn model [6, 9] have been extensively studied in recent years.

These models are simple enough to allow analytical investigations, and rich enough to illustrate most features of nonequilibrium dynamics referred to as aging. The latter include the scaling properties of the correlation and response functions of various observables, and of their fluctuation-dissipation ratios. In this section we recall how the temporal evolution of these two-time quantities can be derived within the master-equation formalism of section 2.

We consider a dynamical urn model of either class, in the thermodynamic limit. For definiteness, our observable will be the local density of balls, defined as the fluctuating population $N_1(t)$ of box number 1, say. The second of the sum rules (2.16) ensures that the mean density $\langle N_1(t)\rangle = \rho$ is constant in time. Its fluctuations $N_1(t) - \rho$ exhibits, however, nontrivial dynamical properties that can be characterized by two-time correlation and response functions, to be defined below. Hereafter, time variables will be such that $0 \leqslant s \leqslant t$. The earlier time s is referred to as the waiting time, the later time t as the observation time.

3.1. Correlation function

The (local, connected) two-time correlation function of the density is defined as

$$C(t, s) = \langle N_1(s)N_1(t)\rangle - \rho^2.$$

This can be recast as

$$C(t, s) = \sum_{k\geqslant 1} k\, \gamma_k(t, s) - \rho^2$$

where the quantities

$$\gamma_k(t, s) = \sum_{j\geqslant 1} j\, f_j(s)\, \mathcal{P}\{N_1(t) = k \mid N_1(s) = j\}$$

assume the initial values

$$\gamma_k(s, s) = k\, f_k(s)$$

at $t = s$. Their temporal evolution for $t \geqslant s$ is given by a master equation similar to (2.14):

$$\frac{\partial \gamma_k(t, s)}{\partial t} = \mu_{k+1}(t)\gamma_{k+1}(t, s) + \lambda_{k-1}(t)\gamma_{k-1}(t, s) - \big(\mu_k(t) + \lambda_k(t)\big)\gamma_k(t, s) \qquad (k \geqslant 1)$$

$$(3.1a)$$

$$\frac{\partial \gamma_0(t, s)}{\partial t} = \mu_1(t)\gamma_1(t, s) - \lambda_0(t)\gamma_0(t, s). \qquad\qquad (3.1b)$$

The rates $\lambda_k(t)$ and $\mu_k(t)$, defined in (2.15), only depend on the $f_k(t)$, so that (3.1) are linear equations for the $\gamma_k(t, s)$.

3.2. Response function

The (local) response function measures the influence on the mean density in box number 1 of a perturbation in the canonically conjugate variable, i.e. the local chemical potential acting on the same box.

Suppose that box number 1 is subjected to a small time-dependent chemical potential $\alpha_1(t)$, so that the total Hamiltonian of the system is now

$$\mathcal{H}(\{N_i\}) = \sum_{i=1}^{M} E(N_i) + \alpha_1(t) N_1.$$

The mean density in box number 1 reads

$$\langle N_1(t) \rangle = \rho + \int_0^t R(t,s)\, \alpha_1(s)\, \mathrm{d}s + \cdots$$

where only the term linear in $\alpha(s)$ has been written explicitly, and causality has been used. The kernel of the linear response

$$R(t,s) = \frac{\delta \langle N_1(t) \rangle}{\delta \alpha_1(s)}$$

is called the two-time response function. The temporal evolution of this function can again be determined by means of the master-equation formalism.

3.3. Fluctuation-dissipation theorem: equilibrium versus nonequilibrium behaviour

Dynamical urn models, like any statistical mechanical model, have a fast convergence to equilibrium, with a finite relaxation time τ_{eq}, in their high-temperature (disordered) phase. If the earlier time exceeds the relaxation time ($s \gg \tau_{\mathrm{eq}}$), the system is at equilibrium. One-time quantities assume their equilibrium values. Two-time quantities, such as the correlation and response functions, are invariant under time translations:

$$C(t,s) = C_{\mathrm{eq}}(\tau) \qquad R(t,s) = R_{\mathrm{eq}}(\tau) \tag{3.2}$$

where

$$\tau = t - s \geqslant 0.$$

Furthermore, these quantities are related by the fluctuation-dissipation theorem

$$R_{\mathrm{eq}}(\tau) = -\frac{1}{T} \frac{\mathrm{d}C_{\mathrm{eq}}(\tau)}{\mathrm{d}\tau}. \tag{3.3}$$

Some many-urn models exhibit a more interesting nonequilibrium behaviour at lower temperatures, where the relaxation time τ_{eq} becomes either very large or infinite, so that the waiting time s and the observation time t can be much smaller than τ_{eq}.

In such a nonequilibrium situation, both time-translation invariance (3.2) and the fluctuation-dissipation theorem (3.3) are violated in general. It has become customary [13–16] to characterize departure from equilibrium by the fluctuation-dissipation ratio

$$X(t,s) = T \frac{R(t,s)}{(\partial C(t,s))/(\partial s)}. \tag{3.4}$$

In general, this dimensionless quantity depends on both times s and t and on the observable under consideration. It often exhibits a nontrivial scaling behaviour in the two-time plane, which is viewed as one of the salient features of aging.

The definition (3.4) can be rephrased in terms of an effective temperature [14]:

$$R(t, s) = \frac{1}{T_{\text{eff}}(t, s)} \frac{\partial C(t, s)}{\partial s} \qquad T_{\text{eff}}(t, s) = \frac{T}{X(t, s)}.$$

At equilibrium, the fluctuation-dissipation theorem is valid, so that $X(t, s) = 1$ and $T_{\text{eff}}(t, s) = T$. For a nonequilibrium situation started from a disordered initial state, such as (2.10), (2.11), the effective temperature is expected to lie somewhere between the formal initial temperature $T(0) = \infty$ and the final one, $T(\infty) = T$, i.e.

$$0 \leqslant X(t, s) \leqslant 1.$$

These inequalities are indeed obeyed in all the known cases.

4. The backgammon model

The backgammon model is a simple example of a system which exhibits slow relaxation due to entropy barriers [7]. This mean-field many-urn model belongs to the Ehrenfest class, with Metropolis dynamics, and the following choice of an energy function:

$$E(N_i) = -\delta(N_i, 0).$$

The Hamiltonian (2.2) is therefore equal to minus the number of empty boxes, so that balls tend to condense in fewer and fewer boxes as time passes, at least at low temperature.

4.1. Statics

The equilibrium occupation probabilities read

$$f_{k,\text{eq}} = \rho\, e^{-\Lambda_{\text{eq}}} \frac{\Lambda_{\text{eq}}^{k-1}}{k!} \times \begin{cases} 1 & (k \geqslant 1) \\ e^\beta & (k = 0). \end{cases}$$

The parameter Λ_{eq}, equal to z_s in the formalism of section 2, represents the mean population of a nonempty box. It is related to temperature and density by

$$e^\beta = 1 + \left(\frac{\Lambda_{\text{eq}}}{\rho} - 1 \right) e^{\Lambda_{\text{eq}}}$$

implying

$$\Lambda_{\text{eq}} \approx \beta - \ln \frac{\beta}{\rho}$$

at low temperature. The roughly linear divergence of Λ_{eq} is the signature of a zero-temperature condensation transition. The density plays no particular role in this phase transition, so that we set $\rho = 1$ in the sequel.

The slow energy relaxation at low temperature can be understood, at least qualitatively, in terms of the concept of entropy barrier, introduced for the Ehrenfest model. The late stages of the dynamics indeed consist in emptying boxes whose typical population is Λ_{eq}. The formula (1.3) suggests the scaling law $\ln \tau_{\text{eq}} \sim \Lambda_{\text{eq}} \sim \beta$. The relaxation time of the energy is indeed known [12] to read

$$\tau_{\text{eq}} \approx \frac{\Lambda_{\text{eq}} - 1}{\Lambda_{\text{eq}}} I(\Lambda_{\text{eq}}) \approx \frac{e^{\Lambda_{\text{eq}}}}{\Lambda_{\text{eq}}} \left(1 + \frac{1}{\Lambda_{\text{eq}}^2} + \cdots \right) \approx \frac{e^\beta}{\beta^2} \left(1 + \frac{2\ln\beta + 1}{\beta} + \cdots \right) \qquad (4.1)$$

at low temperature, with the other characteristic times remaining of order Λ_{eq}. The function

$$I(\Lambda) = \int_0^1 dz\, \frac{e^{\Lambda z} - 1}{z} = \text{Ei}(\Lambda) - \ln\Lambda - \gamma$$

where Ei is the exponential integral and γ is Euler's constant, plays a central role in the analysis of the low-temperature regime.

4.2. Nonequilibrium dynamics: logarithmic relaxation

In the present case, the master equation (2.14) for the occupation probabilities reads

$$\frac{\mathrm{d}f_k(t)}{\mathrm{d}t} = \frac{k+1}{\Lambda(t)}f_{k+1}(t) + f_{k-1}(t) - \left(1 + \frac{k}{\Lambda(t)}\right)f_k(t) \qquad (k \geqslant 2) \qquad (4.2a)$$

$$\frac{\mathrm{d}f_1(t)}{\mathrm{d}t} = \frac{2}{\Lambda(t)}f_2(t) + \mu(t)f_0(t) - 2f_1(t) \qquad (4.2b)$$

$$\frac{\mathrm{d}f_0(t)}{\mathrm{d}t} = f_1(t) - \mu(t)f_0(t) \qquad (4.2c)$$

with

$$\frac{1}{\Lambda(t)} = 1 - (1 - \mathrm{e}^{-\beta})f_0(t) \qquad \mu(t) = \mathrm{e}^{-\beta} + (1 - \mathrm{e}^{-\beta})f_1(t).$$

The parameter $\Lambda(t)$ again represents the mean population of a nonempty box in the low-temperature regime, while the energy density reads

$$E(t) = -f_0(t) \approx -1 + \frac{1}{\Lambda(t)}.$$

The nonequilibrium regime of interest, called the alpha regime in the context of the physics of glasses, corresponds to low temperature and long times. It is natural to investigate the slow relaxation dynamics in this regime by means of an adiabatic decoupling, similar to the Born-Oppenheimer scheme, between the slow evolution of $\Lambda(t)$—due to the emptying of new boxes, and generalizing the timescale τ_{eq}—and the fast evolution of the other degrees of freedom, which corresponds to the diffusive motion of balls between nonempty boxes and generalizes the relaxation times of order Λ_{eq}.

This decoupling scheme can be implemented in several ways. An analytical approach, extensively used in [12], is based on the asymptotic analysis of formal integral representations of the solution to the master equation (4.2). It amounts to neglecting only exponentially small terms, either in Λ or in Λ_{eq}. This approach shows that $\Lambda(t)$ follows an effective Markovian dynamics

$$\frac{\mathrm{d}\Lambda}{\mathrm{d}t} \approx \frac{1}{I(\Lambda(t))}\left[1 - \frac{(\Lambda(t) - 1)\mathrm{e}^{\Lambda(t)}}{(\Lambda_{\mathrm{eq}} - 1)\mathrm{e}^{\Lambda}_{\mathrm{eq}}}\right] \qquad (4.3)$$

throughout the alpha regime ($\Lambda(t)$ and Λ_{eq} both large).

At zero temperature, the term in square brackets in (4.3) is equal to unity, so that

$$\Lambda(t) \approx \ln t + \ln \ln t + \frac{\ln \ln t - 2}{\ln t} + \cdots.$$

The same logarithmic law is present at finite temperature in the aging regime, namely for $\Lambda(t) \ll \Lambda_{\mathrm{eq}}$, i.e. $t \ll \tau_{\mathrm{eq}}$. It is the nonequilibrium counterpart of the equilibrium relationship (4.1) between τ_{eq} and Λ_{eq}.

4.3. Nonequilibrium dynamics: two-time quantities

The two-time quantities defined in section 3 have been studied for both local energy and density fluctuations, by means of the above adiabatic approach [12]. The main results are as follows.

Throughout the alpha regime, the two-time correlation and response functions exhibit a leading power-law scaling, with prefactors which depend logarithmically on both times s and t.

Consider for definiteness zero-temperature dynamics. The energy and density correlation functions read

$$C_E(t, s) \approx \frac{\Phi(s)}{\Phi(t)} \qquad C_D(t, s) \approx (\Lambda(s) - 1) \frac{\Lambda(t)\Phi(s)}{\Lambda(s)\Phi(t)}$$

with

$$\Phi(t) = \exp\left(\int_1^{\Lambda(t)} \frac{d\Lambda}{\Lambda} \frac{\Lambda^2 e^{-\Lambda} I(\Lambda) + 1}{\Lambda^2 e^{-\Lambda} I(\Lambda) + 1 - \Lambda} \right)$$

$$= \sqrt{t} \, \ln t \left(1 + \frac{\ln \ln t - 2}{\ln t} + \cdots \right)$$

so that their leading long-time behaviour is

$$C_E(t, s) \approx \sqrt{\frac{s}{t}} \frac{\ln s}{\ln t} \qquad C_D(t, s) \approx \sqrt{\frac{s}{t}} \ln s.$$

Similar scaling expressions can be obtained for the local energy and density response functions $R_E(t, s)$ and $R_D^\pm(t, s)$. It turns out that two density response functions have to be considered, as a peculiarity of Metropolis dynamics, because the acceptance rate (2.12) is not differentiable (it has a cusp) at $\Delta \mathcal{H} = 0$. The associated fluctuation-dissipation ratios are independent of the observation time t throughout the alpha regime, provided $\tau = t - s \gg 1$ is a macroscopic time.

For a large waiting time s, one has

$$X_E(t, s) \approx 1 - \frac{2}{\Lambda(s)^2} - \frac{4}{\Lambda(s)^3} - \frac{6}{\Lambda(s)^4} + \cdots$$

$$X_D^+(t, s) \approx 1 - \frac{2}{\Lambda(s)^2} - \frac{2}{\Lambda(s)^3} - \frac{12}{\Lambda(s)^4} + \cdots$$

$$X_D^-(t, s) \approx 1 - \frac{3}{\Lambda(s)^2} - \frac{3}{\Lambda(s)^3} - \frac{15}{\Lambda(s)^4} + \cdots.$$

The fluctuation-dissipation theorem is thus asymptotically restored in the long-time regime, with a leading logarithmic correction in $1/(\ln s)^2$, which depends both on the observable (energy versus density fluctuations) and on microscopic details of the model (Metropolis versus heat-bath dynamics).

5. The zeta urn model

The static zeta urn model has been introduced [17] as a mean-field model of discretized quantum gravity. Its dynamics has been subsequently investigated [6, 9]. The zeta urn model belongs to the monkey class, with mean-field heat-bath dynamics, defined by the energy function

$$E(N_i) = \ln(N_i + 1). \tag{5.1}$$

5.1. Statics

The logarithmic confining potential (5.1) is such that the Boltzmann weight

$$p_{N_i} = (N_i + 1)^{-\beta}$$

falls off as a power law, with a temperature-dependent exponent. The corresponding generating series has a power-law singularity at $z_c = 1$, of the form $P_{sg}(z) \sim (1 - z)^{\beta-1}$, allowing a condensation transition for finite values of temperature and density. At low enough temperature

($\beta > 2$), $P_{sg}(z)$ vanishes more rapidly than linearly as $z \to 1$, so that there is a finite critical density, corresponding to $z_s = z_c = 1$, namely

$$\rho_c = \frac{P'(1)}{P(1)} = \frac{\zeta(\beta - 1) - \zeta(\beta)}{\zeta(\beta)}$$

where ζ denotes Riemann's zeta function, which also enters (5.2), hence the name of the model, proposed in [6]. The static phase diagram of the model is as follows [17].

Fluid phase ($\rho < \rho_c$).   In this low-density phase we have $z_s < 1$, so that the equilibrium occupation probabilities (2.9) decay exponentially. The density goes continuously to its critical value ($\rho \to \rho_c^-$) as $z_s \to 1^-$.

Criticality ($\rho = \rho_c$).   Along the critical line in the temperature–density plane, the equilibrium occupation probabilities

$$f_{k,eq} = \frac{(k + 1)^{-\beta}}{\zeta(\beta)} \tag{5.2}$$

obey a zeta distribution, with a temperature-dependent exponent. More generally, critical exponents and other universal quantities depend continuously on temperature along the critical line. The variance of the population of box number 1

$$C_{eq} = \sum_{k \geqslant 0} k^2 f_{k,eq} - \rho_c^2 = \frac{\zeta(\beta)\zeta(\beta - 2) - \zeta(\beta - 1)^2}{\zeta(\beta)^2} \tag{5.3}$$

is finite for $\beta > 3$ (regular critical regime), while it is infinite in the strong-fluctuation critical regime ($2 < \beta < 3$).

Condensed phase ($\rho > \rho_c$).   A macroscopic condensate appears in this high-density phase. Indeed, the static configurations are such that the critical occupation probabilities (5.2) still hold for all boxes but one, in which an extensive number of balls, of order $N - M\rho_c = M(\rho - \rho_c)$, is condensed.

5.2. Nonequilibrium critical dynamics ($\rho = \rho_c$)

Nonequilibrium properties of the zeta urn model have been studied recently [6], both at criticality and in the condensed phase, pursuing earlier investigations [9]. The most salient results are as follows.

Consider the situation of regular criticality ($\beta > 3$), with a disordered initial state such as (2.11). For any large but finite time t, the system looks critical, i.e. the occupation probabilities $f_k(t)$ have essentially converged toward their equilibrium values (5.2), for $k \ll t^{1/2}$, while for $k \gg t^{1/2}$ the system still looks disordered, i.e. the $f_k(t)$ fall off very fast. At a quantitative level, this crossover behaviour is described by the scaling law

$$f_k(t) \approx f_{k,eq} F(u) \quad u = k\,t^{-1/2}. \tag{5.4}$$

The scaling function $F(u)$ is determined by explicitly solving the differential equation describing the continuum limit of the master equation (2.14). Among other consequences, the variance of the population of box number 1 is found to converge to its equilibrium value (5.3) as a power law:

$$C(t, t) = \langle N_1(t)^2 \rangle - \rho_c^2 \approx C_{eq} - \frac{2^{3-\beta}\,t^{-(\beta-3)/2}}{(\beta - 3)\,\Gamma((\beta + 1)/2)\,\zeta(\beta)}. \tag{5.5}$$

The two-time local density correlation and response functions have been investigated along the same lines [6]. In the nonequilibrium scaling regime ($s, t \gg 1$), these two-time quantities are found to scale as

$$C(t, s) \approx s^{-(\beta-3)/2} \, \Phi(x)$$
$$\frac{\partial C(t, s)}{\partial s} \approx s^{-(\beta-1)/2} \, \Phi_1(x) \qquad (5.6)$$
$$R(t, s) \approx s^{-(\beta-1)/2} \, \Phi_2(x)$$

where the dimensionless time ratio

$$x = t/s \geqslant 1$$

is the temporal analogue of an aspect ratio. As a consequence, the fluctuation-dissipation ratio $X(t, s)$ only depends on x in the scaling regime:

$$X(t, s) \approx \mathcal{X}(x) = T \frac{\Phi_2(x)}{\Phi_1(x)}.$$

The above results illustrate general predictions on nonequilibrium critical dynamics [18–21]. The exponent of the waiting time s in the first line of (5.6) already appears in (5.5). It is related to the anomalous dimension of the observable under consideration, and would read $(d - 2 + \eta)/z_c$ for a d-dimensional ferromagnet, with η being the equilibrium correlation exponent and z_c the dynamical critical exponent. The scaling functions $\Phi(x)$, $\Phi_{1,2}(x)$ are universal up to an overall multiplicative constant, and they obey a common power-law fall-off in $x^{-\beta/2}$. The latter exponent is not related to exponents pertaining to usual (equilibrium) critical dynamics. It would read $-\lambda_c/z_c = \Theta_c - d/z_c$ for a ferromagnet, where λ_c is the critical autocorrelation exponent [22] and Θ_c is the critical initial-slip exponent [18].

As a consequence, the dimensionless scaling function $\mathcal{X}(x)$ is universal, and it admits a nontrivial limit value in the regime where both time variables are well separated in the scaling regime:

$$X_\infty = \lim_{s \to \infty} \lim_{t \to \infty} X(t, s) = \mathcal{X}(\infty).$$

Explicit expressions for the above scaling functions have been derived, using a spectral decomposition in Laguerre polynomials. Let us mention that the limit fluctuation-dissipation ratio,

$$X_\infty = \frac{\beta + 1}{\beta + 2} \quad (\beta > 3)$$

lies in an unusually high range ($4/5 < X_\infty < 1$) for a critical system. Indeed, statistical mechanical models such as ferromagnets are observed to have $0 < X_\infty \leqslant 1/2$ at their critical point. The upper bound $X_\infty = 1/2$, corresponding to the mean-field situation [20], is also observed in a range of simpler models [13, 15, 16, 19].

5.3. Nonequilibrium condensation dynamics ($\rho > \rho_c$)

Consider now a disordered initial condition such as (2.11) with $\rho > \rho_c$, so that the system will build up a macroscopic condensate. The main results concerning this situation are as follows [6].

The growth of the condensate exhibits a scaling regime for $1 \ll t \ll M^2$, while finite-size effects become dominant for later times. In this scaling regime, the occupation probabilities $f_k(t)$ have essentially converged toward their equilibrium values (5.2) for $k \ll t^{1/2}$, while each box has a small probability, of order $t^{-1/2}$, to be part of the growing condensate, and to contain

a large number of balls, of order $t^{1/2}$. This picture can again be turned to a scaling law, which is of the form

$$f_k(t) \approx \frac{F(u)}{t} \quad u = k\, t^{-1/2}.$$

In contrast to the scaling function $F(u)$ of (5.4), neither the present scaling function nor those entering (5.7) can be expressed in closed form, because the continuum limit of the master equation (2.14) in the condensed phase involves a biconfluent Heun differential equation, whose solutions are not known explicitly.

The investigation of the scaling regime of condensation dynamics can be extended to two-time quantities [6], which are found to scale as

$$
\begin{aligned}
C(t, s) &\approx (\rho - \rho_c) s^{1/2}\, \Phi(x) \\
\frac{\partial C(t, s)}{\partial s} &\approx (\rho - \rho_c) s^{-1/2}\, \Phi_1(x) \\
R(t, s) &\approx (\rho - \rho_c) s^{-1/2}\, \Phi_2(x) \\
X(t, s) &\approx \mathcal{X}(x) = T\, \frac{\Phi_2(x)}{\Phi_1(x)}.
\end{aligned}
\tag{5.7}
$$

This condensation dynamics exhibits many peculiarities with respect to usual phase-ordering dynamics [23], which takes place, for example, if a ferromagnet is quenched below its critical temperature. In the latter case, domain growth and phase separation take place in a statistically homogeneous way, at least for an infinite system. In contrast, in the present situation, condensation takes place in a very inhomogeneous fashion, as fewer and fewer boxes are involved.

Among unusual features of two-time quantities, let us emphasize that the scaling functions $\Phi(x)$, $\Phi_{1,2}(x)$ assume finite values, both at coinciding times ($x = 1$) and in the limit of large time separations ($x = \infty$). Furthermore, the limit fluctuation-dissipation ratio $X_\infty = \mathcal{X}(\infty)$ depends continuously on temperature throughout the condensed phase ($\beta > 2$, i.e. $T < 1/2$), and it vanishes only as

$$X_\infty = T^{1/2} - \frac{T^{3/4}}{4} + \cdots$$

at low temperature, while coarsening systems are known [24] to have identically $X_\infty = 0$ throughout their low-temperature phase.

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